ISRAEL JOURNAL OF MATHEMATICS, Vol. 25, 1976

ON A CLASS OF VALUATION FIELDS INTRODUCED BY A. ROBINSON[†]

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ABSTRACT

It is shown that the nonarchimedean valuation fields ${}^{\rho}R$ introduced by A. Robinson are not only complete but are also spherically complete. Furthermore, it is shown that to every normed linear space over the reals there exists a nonarchimedean normed linear space ${}^{\rho}E$ over ${}^{\rho}R$ in the sense of Monna which is spherically complete and extends E.

1. Introduction

The term nonarchimedean analysis usually refers to analysis which involves fields which admit a nonarchimedean valuation such as the p-adic number fields. By nonarchimedean functional analysis we mean the theory of linear spaces over nonarchimedean valuation fields endowed with a nonarchimedean norm in the sense of Monna.

The creation of nonstandard analysis in the early sixties by Abraham Robinson provides us with a new technique of studying problems of analysis with the help of special nonarchimedean fields. That there exists also a close link between nonstandard analysis and nonarchimedean analysis was first shown by Abraham Robinson. This was established by showing that every nonstandard number system gives rise to a special nonarchimedean valuation field ${}^{\rho}R$ in which the classical nonarchimedean power series field, first studied in detail by Levi-Civitâ, can be imbedded.

The purpose of the present note is twofold. First of all we shall show that the fields ${}^{\rho}R$ share with the power series fields the property of being maximal, or what is the same, are spherically complete. Secondly, we show that there is also a close link between nonstandard analysis and nonarchimedean functional

[†] Dedicated to the memory of A. Robinson.

[‡] Work on this paper was supported in part by NSF Grant MPS 74–17845

analysis. This is accomplished by showing that every normed linear space E over the reals can be embedded in a nonarchimedean normed linear space ${}^{\rho}\hat{E}$ over ${}^{\rho}R$ which is spherically complete.

2. The fields ${}^{\rho}R$ introduced by A. Robinson

For the terminology, definitions and results about nonarchimedean fields not explained in this paper we refer the reader to [9] and [12]. In addition, the reader is referred to either of the references [6], [8], [10], and [12] for the basic notions and techniques of nonstandard analysis used below.

For a proper understanding of the paper we shall first recall a number of definitions and results from the theory of nonarchimedean fields which play an important role in the present paper.

In the following, R will always denote the set of real numbers. For the sake of convenience we shall adopt the convention that a symbol "∞" is added to R obeying the rules $x + \infty = \infty + x$ for all $x \in R$, $\infty + \infty = \infty$ and $x < \infty$ for all $x \in R$.

A mapping v of a totally ordered commutative field K into $R \cup \{\infty\}$ is called a nonarchimedean valuation whenever it satisfies the following conditions:

(2.1) $v(0) = \infty$ and $v(x) \in R$ for all $x \in K$, $x \neq 0$.

(2.2) For all $x, y \in K$, $v(xy) = v(x) + v(y)$.

(2.3) For all $x, y \in K$, $v(x + y) \geqq \min(v(x), v(y))$.

A valuation is called trivial whenever $v(x) = 0$ for all $x \in K$ and $x \neq 0$. All the valuations which occur in this paper are non-trivial and real-valued.

From (2.2) it follows easily that $v(1) = v(-1) = 0$ and $v(x) = v(-x) = -v(1/x)$ for all non-zero $x \in K$.

The set $0_K = \{x : v(x) \geqq 0\}$ is an integral domain, the ring of integers of K. The set $I_K = \{x : v(x) > 0\}$ is a maximal ideal in 0_K, the valuation ideal of K. The quotient field $0_K / I_K$ is called the residue class field of K. The set $U_K = \{x : v(x) = 0\}$ is a multiplicative subgroup of 0_K, the group of units of K. Finally (2.2) shows that the set of values taken on by v is in general a subgroup of the additive group of the reals, the value group of K, which will be denoted by V_K.

The definition $|x|_v = \exp(-v(x))$, $x \in K$, where "exp" denotes the exponential function and where $\exp(-\infty)$ is to be interpreted as zero, turns K into a metric space. This metric has the following properties.

(2.4) $|0|_v = 0$, $|1|_v = 1$ and $|x_v| > 0$ for all $x \in K$ and $x \neq 0$.

(2.5) $|-x|_v = |x|_v$ for all $x \in K$.

(2.6) $|xy|_v = |x|_v |y|_v$ for all $x, y \in K$.

(2.7) $|x+y|_v \leqq \max(|x|_v, |y|_v) \leqq |x|_v + |y|_v$ for all $x, y \in K$.

From (2.7) it follows that $|x-y|_v > |y-z|_v$ implies $|x-z|_v = |x-y|_v$. Hence, the space $(K, |\cdot|_v)$ is a metric space in which every triangle is isosceles. For this reason property (2.7) is often referred to as the ultrametric inequality.

A set of the form $\{x : |x - x_0|_v \leqq r\}$, where $x_0 \in K$ and $0 < r \in R$, is called a closed ball. Any two closed balls are either disjoint or one is contained in the other. The center of a ball is not uniquely determined because of the ultrametric inequality.

A nonarchimedean valuation field K is called complete if the associated metric space $(K, |\cdot|_v)$ is a complete metric space. In addition to the notion of completeness a stronger notion, that of spherical completeness, plays a very important role in the theory of nonarchimedean analysis. We shall now present the precise definition of this notion and discuss some of its consequences.

(2.8) Definition. A nonarchimedean valuation field is called spherically complete whenever every non-empty family of closed balls with the property that every pair has a non-empty intersection has already a non-empty intersection.

It is easy to see that a nonarchimedean valuation field is spherically complete if every decreasing sequence of closed balls has a non-empty intersection. The reader should recall here that we are only considering real-valued valuations.

A spherically complete nonarchimedean field is always complete. Indeed, if $x_n \in K$ $(n = 1, 2, \cdots)$ is a $|\cdot|_v$-Cauchy sequence, then the decreasing sequence of closed balls $\{x : |x - x_n|_v \leqq \max_{k \geqq n} |x_k - x_{k+1}|_v\}$ has a non-empty intersection, which can be shown to be the $|\cdot|_v$-limit of the sequence $\{x_n\}$.

Following Ostrowski (see [1]) we define.

(2.9) Definition. A mapping $i \rightarrow x_i$ of a well-ordered set I without a last element into K is called pseudo-convergent whenever $i, j, k \in I$ and $i < j < k$ implies $|x_k - x_j|_v < |x_j - x_i|_v$. An element $x_0 \in K$ is called a pseudo-limit of a pseudo-convergent sequence $\{x_i : i \in I\}$ whenever $|x_0 - x_i|_v = |x_{i+1} - x_i|_v$ for all $i \in I$. A pseudo-convergent sequence is said to be convergent if it possesses a pseudo-limit.

If $\{x_i : i \in I\}$ is pseudo-convergent, then $|x_j - x_i| = |x_{i+1} - x_i|_v$ for all $i, j \in J$ satisfying $i < j$. The pseudo-limit of a pseudo-convergent sequence is by no means unique. If x_0 is a pseudo-limit of $\{x_i : i \in I\}$, then y is pseudo-limit of $\{x_i\}$ if and only if it can be written in the form $y = x_0 + x$, where $x \in K$ satisfies $|x|_v < |x_{i+1} - x_i|_v$ for all $i \in I$.

Concerning the notion of spherical completeness we have the following results.

(2.10) THEOREM. *A nonarchimedean valuation field K is spherically complete if and only if every pseudo-convergent sequence of K is convergent.*

PROOF. If K is spherically complete and $\{x_i : i \in I\}$ a pseudo-convergent sequence in K, then it follows immediately that the family of balls $B_i = \{x : |x - x_i|_v \leqq |x_i - x_{i+1}|_v\}$ has the property that every pair has a non-empty intersection. Hence, by the spherical completeness of K there exists an element $x_0 \in K$ such that $x_0 \in B_i$ for all $i \in I$. It is then easy to see that x_0 is a pseudo-limit of $\{x_i : i \in I\}$.

Conversely, assume that $\{B\}$ is a non-empty family of balls of K such that every pair of elements of $\{B\}$ has a non-empty intersection. $\{B\}$ is totally-ordered by inclusion. Let $r_0 = \inf(r(B) : B \in \{B\})$, where $r(B)$ is the radius of B. If there exists an element $B_0 \in \{B\}$ with $r(B_0) = r_0$, then $B_0 \subset B$ for all $B \in \{B\}$ and the proof is finished. If this is not the case, however, then there exists a decreasing sequence $\{B_n\} \subset \{B\}$ with $r(B_n)$ decreasing to r_0 and a sequence $\{x_n\}$ of elements of K such that $x_n \in B_n$ and $x_n \not\in B_{n+1}$ for all $n = 1, 2, \cdots$. It is easy to see that the sequence $\{x_n\}$ is pseudo-convergent and that a pseudo-limit of $\{x_n\}$ is contained in all the balls of $\{B\}$. This completes the proof of the theorem.

In [1], Kaplansky has shown that the notion of spherical completeness is connected with a maximality property of nonarchimedean fields introduced by F. K. Schmidt and first published by W. Krull in [2].

(2.11) DEFINITION. A nonarchimedean field K is called maximal if K does not admit a proper extension to a nonarchimedean valuation field K' such that K and K' have the same valuation group and the same residue class field.

From Theorem 2.10 and a result of Kaplansky [1] the following theorem follows.

(2.12) THEOREM. *Let K be a nonarchimedean valuation field. Then the following conditions are equivalent.*

i) *K is spherically complete.*

ii) *Every pseudo-convergent sequence in K is convergent.*

iii) *K is maximal.*

The equivalence of (ii) and (iii) is due to Kaplansky [1], theorem 4.

We shall now turn our attention to the property of the special nonarchimedean fields ${}^{\rho}R$ introduced by A. Robinson [11], see also [12].

Let $\mathfrak{M}$ be a superstructure in the sense of [5] based on a set of individuals sufficiently large to contain R and let ${}^*\mathfrak{M}$ be an ultrapower enlargement of $\mathfrak{M}$. Then by ${}^*R$ we shall denote the set of hyperreals of ${}^*\mathfrak{M}$ or the nonstandard real number system determined by $\mathfrak{M}$.

For a proper understanding of what is to follow the reader should know that such enlargements are sequentially comprehensive in the sense that if $\{a_n\}$, $n = 0, 1, 2, \cdots$ is an external sequence of entities of ${}^*\mathfrak{M}$ of the same rank, then there exists an internal sequence $\{b_n\}$ in ${}^*\mathfrak{M}$, where n now runs over ${}^*N$, the set of natural numbers of ${}^*\mathfrak{M}$, such that $a_n = b_n$ for all finite or standard $n = 0, 1, 2, \cdots$. From this fact, the following basic lemma follows (see [12], ch. 3).

(2.13) LEMMA. *Let $\{a_n\}$, $n = 0, 1, 2, \cdots$ be a strictly increasing (decreasing) sequence of infinitely large (infinitely small) positive numbers. Then there exists an infinitely large (infinitely small) positive number b such that $a_n < b$ $(a_n > b)$ for all $n = 0, 1, 2, \cdots$.*

Now let $0 < \rho$ be an arbitrary but fixed positive infinitesimal in ${}^*R$. We then define subsets ${}^{\rho}M_0$ and ${}^{\rho}M_1$ by

(2.14)
$$
\begin{aligned}
{}^{\rho}M_0 &= \{x : x \in {}^*R \text{ and } |x| < \rho^{-n} \text{ for some positive integer } n\}, \text{ and}\\
{}^{\rho}M_1 &= \{x : x \in {}^*R \text{ and } |x| < \rho^{n} \text{ for all finite positive integers } n = 0, 1, 2, \cdots\}.
\end{aligned}
$$

Evidently, ${}^{\rho}M_1$ is contained in ${}^{\rho}M_0$, ${}^{\rho}M_1$ is a subset of the set of infinitesimals M_1 of ${}^*R$ and the set of finite numbers M_0 of ${}^*R$ is contained in ${}^{\rho}M_0$. The set ${}^{\rho}M_0$ inherits from ${}^*R$ the algebraic and order operations. From this very definition it follows immediately that ${}^{\rho}M_0$ is an integral domain and that ${}^{\rho}M_1$ is a unique maximal order ideal in ${}^{\rho}M_0$. Hence, the quotient ${}^{\rho}R = {}^{\rho}M_0/{}^{\rho}M_1$ is a totally-ordered field.

The canonical mapping of ${}^{\rho}M_0$ onto ${}^{\rho}R$ with kernel ${}^{\rho}M_1$ will be denoted by ${}^{\rho}\mathrm{st}$. Then ${}^{\rho}\mathrm{st}$ is an order preserving homomorphism. Since ${}^{\rho}M_1$ contains only one single standard number, namely zero, ${}^{\rho}\mathrm{st}$ maps R, the real numbers in a one-to-one manner onto a subfield of ${}^{\rho}R$. From now on we shall identify ${}^{\rho}\mathrm{st}(R)$ with R.

Recall that the set of finite elements of ${}^*R$ is denoted by M_0 ($a \in M_0$ whenever there exists an element $r \in R$ with $|a| < r$), the set of infinitesimals of

$^*R$ by M_1 ($h \in M_1$ whenever $|h| < \varepsilon$ for all $0 < \varepsilon \in R$), M_1 is the unique maximal ideal of M_0 and the algebraic and order homomorphism of M_0 onto R with kernel M_1 is denoted by "st" and which is called the standard part operation. Furthermore, $x =_1 y$ means that $x - y \in M_1$, and so $\mathrm{st}(a) =_1 a$ for all $a \in M_0$. We shall now introduce the following additional notation. If $x, y \in {}^*R$, then $x =_\rho y$ will mean that $x - y \in {}^\rho M_1$. Then for any $a \in {}^\rho M_0$ we have ${}^\rho\mathrm{st}(a) = {}_\rho a$.

Since $^*R$ is a nonstandard model of R it follows from the transfer principle that all the elementary functions of R extend uniquely to $^*R$ preserving their properties as far as they occur in the list of true statements of the superstructure $\mathfrak{M}$. Hence, for every $a \in {}^*R$, $a > 0$, $\log_\rho a$, where $\log_\rho$ stands for the logarithmic function with base ρ, is well-defined.

Assume now that $x \in {}^\rho M_0$ and $h \in {}^\rho M_1$. Then it follows immediately from the definition of ${}^\rho M_0$ that the numbers $\log_\rho |x|$ and $\log_\rho |x + h|$ are finite numbers of $^*R$. Furthermore, $\log_\rho |x + h| - \log_\rho |x| = \log_\rho |1 + h/x| = (\log |1 + h/x|)/\log\rho$ shows, using the facts $|\log \rho|$ is infinitely large and $\log |1 + h/x|$ is infinitely small, that $\log_\rho |x| =_1 \log_\rho (|x + h|)$. Hence, $\mathrm{st}(\log_\rho |x + h|) = \mathrm{st}(\log_\rho |x|)$ for all $x \in {}^\rho M_0$ and $h \in {}^\rho M_1$. Consequently, by putting $v_\rho(0) = \infty$ and $v_\rho(a) = \mathrm{st}(\log_\rho (|x|))$, where $a = {}^\rho\mathrm{st}(x)$, $x \in {}^\rho M_o$, we define a unique function v_ρ on ${}^\rho R$. We have now the following important result due to A. Robinson [11]. For the proof we refer to [11] or [12].

(2.15) THEOREM. *v_ρ defines a non-trivial nonarchimedean valuation on ${}^\rho R$.*

It is obvious from the definition of v_ρ that the value group is the additive group of the reals. The residue class field is more difficult to determine. The valuation ring $0_{{}^\rho R}$ of ${}^\rho R$ consists of all the elements $a \in {}^\rho R$ with the property that for each $x \in {}^\rho M_0$ satisfying ${}^\rho\mathrm{st}(x) = a$ there exists an infinitesimal $h > 0$ such that $|x| < \rho^{-h}$. The valuation ideal $I_{{}^\rho R}$ of ${}^\rho R$ consists of all the elements $a \in {}^\rho R$ with the property that for each $x \in {}^\rho M_0$ satisfying ${}^\rho\mathrm{st}(x) = a$ there exists a positive standard real number $\varepsilon > 0$ such that $|x| < \rho^\varepsilon$. It is clear from the formula $\log_\rho |x| = \log |x| / \log \rho$ that all the non-zero standard numbers in ${}^\rho R$ are units. Hence, the residue class field $0_{{}^\rho R}/I_{{}^\rho R}$ of ${}^\rho R$ contains R as a subfield. Since the number $\log |\log \rho| / |\log \rho|$ is infinitely small it follows that the element $a = {}^\rho\mathrm{st}(|\log \rho|)$ of ${}^\rho R$ is a unit and so the residue class field of ${}^\rho R$ is a proper extension of R. In fact, it is a totally-ordered nonarchimedean field.

The mapping $|\cdot|_\rho = \exp(-v_\rho(\cdot))$ defines an ultrametric on ${}^\rho R$. It was shown in [11], by A. Robinson, that the metric space $({}^\rho R, |\cdot|_\rho)$ is complete. We shall now improve upon this result by showing that ${}^\rho R$ has the following property.

(2.16) THEOREM. *$({}^{\rho}R, |\cdot|_\rho)$ is spherically complete.*

PROOF. According to Theorem 2.10 we have to show that every pseudo-convergent sequence is convergent. Since the valuation is real-valued we have only to show that every pseudo-convergent sequence $\{a_n\}$, $n = 1, 2, \cdots$ of elements of ${}^{\rho}R$ is convergent.

To this end, we determine first for each $n = 1, 2, \cdots$ an element $x_n \in {}^{\rho}M_0$ such that ${}^{\rho}\mathrm{st}(x_n) = a_n$. Since the sequence $\{a_n\}$ is pseudo-convergent we have that $v_\rho(a_m - a_n) = v_\rho(a_{n+1} - a_n)$ for all $m \geqq n = 1, 2, \cdots$. From the definition of v_ρ it follows that for all $m, n = 1, 2, \cdots$ satisfying $m \geqq n$ we have

$$\log_\rho |x_m - x_n| - \log_\rho |x_{n+1} - x_n| =_1 0. \tag{2.17}$$

We may assume without loss of generality that the sequence $\{x_n\}$ has been extended to an internal sequence over $^*N$ with values in $^*R$, which we shall denote again by $\{x_n\}$, $n \in {}^*N$. Then from (2.17) and Lemma (2.13) we may conclude that for each $n = 1, 2, \cdots$ there exists an infinitely large natural number ω_n and a positive infinitesimal h_n such that the sequence $\{\omega_n\}$, $n = 1, 2, \cdots$ is decreasing and the sequence $\{h_n\}$, $n = 1, 2, \cdots$ is increasing and that

$$\big|\log_\rho |x_m - x_n| - \log_\rho |x_{n+1} - x_n|\,\big| \leqq h_n \quad \text{for all } m \text{ satisfying } m \leqq \omega_n. \tag{2.18}$$

Using again Lemma 2.13 we may conclude that there exists a positive infinitesimal $h_0 > 0$ and an infinitely large natural number ω_0 such that $h_n \leqq h_0$ for all $n = 1, 2, \cdots$ and $\omega_n \geqq \omega_0$ for all $n = 1, 2, \cdots$. Hence,

$$\big|\log_\rho |x_{\omega_0} - x_n| - \log_\rho |x_{n+1} - x_n|\,\big| \leqq h_0$$

for all $n = 1, 2, \cdots$. This implies that $x_{\omega_0} \in {}^{\rho}M_0$ and that $a_0 = {}^{\rho}\mathrm{st}(x_{\omega_0})$ satisfies $v_\rho(a_0 - a_n) = v_\rho(a_{n+1} - a_n)$ for all $n = 1, 2, \cdots$, that is, a_0 is a pseudo-limit of the sequence $\{a_n\}$. This completes the proof of the theorem.

REMARK. The general power series fields L introduced by T. Levi-Cività in [4] (see also [3] and [12]) are fields whose elements are the formal expressions $\Sigma_{k=0}^{\infty} a_k t^{\nu_k}$, where the coefficients a_k and the exponents ν_k are real. The sequence $\{\nu_k\}$ is arranged in increasing order, that is, $\nu_k < \nu_{k+1}$ for all $k = 0, 1, 2, \cdots$ and is assumed to be unbounded. Two such expressions are regarded, by definition, as equal if for each term at^ν which occurs in one but not in the other, $a = 0$. The sum and product are defined in the usual manner, and as one of the results one can show that L is a field under these operations with $1 \cdot t^0 + 0t^1 + 0t^2 + \cdots$ as unit. Furthermore L is totally-ordered by defining an expression to be positive if the first non-vanishing coefficient is positive. We can also define on L a valuation

v_L by setting $v_L(0) = \infty$ and $v_L(\Sigma_0^\infty a_k t^{\nu_k}) = \nu_m$ where m is the first of all the indices of the non-vanishing coefficients. In this case, it is easy to see that the value group is the multiplicative group of the reals and that the real number system is its residue class field. Furthermore, it can easily be shown that L is spherically complete, that is, L is maximal. For a detailed account of the above result about L we refer the reader to [3], [4] and [12]. It was first shown by A. Robinson [11] and [12] that L is isomorphic algebraically and isometrically to a subfield of ${}^\rho R$. This algebraic isometry can be given by means of the following definition. Let $\bar{\rho} = {}^\rho\mathrm{st}(\rho)$ and assign to any formal expression $\Sigma_0^\infty a_k t^{\nu_k}$ the formal infinite ρ-series $\Sigma_0^\infty a_k \bar{\rho}^{\nu_k}$ of ${}^\rho R$. It can be shown that the infinite ρ-series are convergent in ${}^\rho R$ and that the mapping $\Sigma_0^\infty a_k t^{\nu_k} \to \Sigma_0^\infty a_k \bar{\rho}^{\nu_k}$ of L onto ${}^\rho R$ is an algebraic isomorphism which preserves the valuation.

Since both fields L and ${}^\rho R$ are maximal and have different residue class fields ${}^\rho R$ is a proper maximal extension of the Levi-Cività field. The imbedding of L into ${}^\rho R$ can be used to facilitate the definition of functions of L into L and the investigation of certain asymptotic series. For details of this aspect of the fields ${}^\rho R$ we refer to the recent book [12].

For different values of the infinitesimal ρ we generate different maximal fields ${}^\rho R$ which are extensions of R. For a given model ${}^*R$ of R the fields ${}^\rho R$ have the same value group and isomorphic residue class fields. From a result of Kaplansky [1], theorem 7, it follows that for each pair of positive infinitesimals ρ_1, ρ_2 the fields ${}^{\rho_1}R$ and ${}^{\rho_2}R$ are analytically equivalent over R, that is, there exists an isomqrphism of ${}^{\rho_1}R$ onto ${}^{\rho_2}R$ which is value preserving and the identity on R.

3. Nonarchimedean functional analysis

The theory of normed spaces over nonarchimedean valuation fields has its origin in the theory of nonarchimedean distance functions which were introduced by Hausdorff. A metric space (X, d) is called nonarchimedean whenever for every triple of points $x, y, z \in X$ $d(x,y) \leqq \max(d(x,z), d(y,z))$. The connection between nonarchimedean metrics and nonarchimedean valuation fields can be seen by observing that, if (K, v) is a nonarchimedean valuation field, then the metric space $(K, |\cdot|_v)$ is nonarchimedean. Furthermore, if f is a mapping of a non-empty set X into K, then $d(x,y) = \exp(-v(f(x)-f(y))$ is a nonarchimedean semi-metric on X. Every nonarchimedean metric space is zero-dimensional. In fact, it is not difficult to show that every non-zero-dimensional metric space can be given an equivalent nonarchimedean metric. Furthermore, Hausdorff has shown that every metric space is the continuous image of a nonarchimedean metric space.

In the present paper we shall, however, restrict ourselves to the theory of nonarchimedean normed linear spaces in the sense of A. F. Monna [9]. Such spaces considered as metric spaces are particular examples of nonarchimedean metric spaces. We recall their definitions. Let (K, v) be a nonarchimedean valuation field with the metric $|\cdot|_v = \exp(-v(\cdot))$. A linear space E over K is called a nonarchimedean normed linear space if there exists a mapping p of E into the reals which satisfies the following conditions:

i) $p(x) = 0$ if and only if $x = 0$,

ii) for all $a \in K$ and $x \in E$ we have $p(ax) = |a|_v p(x)$,

iii) $p(x + y) \leqq \max(p(x), p(y))$ for all $x, y \in E$.

From these properties it follows immediately that $p(x) \geqq 0$ for all x, $p(x) = p(-x)$ and $p(x + y) = \max(p(x), p(y))$ if $p(x) \neq p(y)$. If the metric space (E, p) is complete, then (E, p) is called a nonarchimedean Banach space.

The theory of nonarchimedean normed linear spaces has been developed extensively and the reader who is interested in the general theory is referred to a recent book by A. F. Monna [9].

The purpose of the present section is to show that, using nonstandard number systems, there exists a close link between the classical theory of normed linear spaces and the theory of nonarchimedean normed linear spaces. In fact, what we shall show is that every normed linear space over the reals can be imbedded in a particular fashion, to be explained later, in a nonarchimedean normed linear space. The construction is closely analogous to the construction of the nonstandard hull of a normed linear space introduced by the author in [5].

Given a normed linear space (E, p) over R or a family of normed linear spaces we shall again denote by $\mathfrak{M}$ a superstructure based on a set of individuals which is sufficiently large as to contain the linear space E or the elements of the spaces of the given family as members. In addition to $\mathfrak{M}$ we consider our ultrapower enlargement ${}^*\mathfrak{M}$ of $\mathfrak{M}$. Then, if (E, p) is an entity of $\mathfrak{M}$ we shall denote by $({}^*E, {}^*p)$ the corresponding entity of ${}^*\mathfrak{M}$ which has the same properties relative to ${}^*\mathfrak{M}$ as (E, p) has to $\mathfrak{M}$ as far as they can be formulated in terms of the language selected to express the properties of $\mathfrak{M}$. To facilitate the discussion we shall first briefly recall the definitions of the nonstandard hull of a normed linear space.

Let (E, p) be a normed linear space and let $({}^*E, {}^*p)$ denote its extension in ${}^*\mathfrak{M}$.

An element $a \in {}^*E$ is called finite whenever ${}^*p(a)$ is a finite number of ${}^*R$. The set of all norm-finite elements of ${}^*E$ will be denoted by p-fin$({}^*E)$. By $\mu_p(0)$ we shall denote the elements of ${}^*E$ who's norm are infinitely small, that is,

$a \in \mu_\rho(0)$ whenever ${}^*\rho(a) =_1 0$. We shall now consider p-fin$({}^*E)$ as a linear space over R. Then $\mu_p(0)$ is a linear subspace of p-fin$({}^*E)$. The quotient space p-fin$({}^*E)/\mu_p(0)$ is called the nonstandard hull of E and is denoted by $\hat{E}$. The canonical mapping of p-fin$({}^*E)$ onto $\hat{E}$ with kernel $\mu_p(0)$ will be denoted by "st_p". Now a norm $\hat{p}$ can be defined by setting $\hat{p}(a) = \mathrm{st}({}^*p(x))$, where $a = \mathrm{st}_p(x)$. Then $(\hat{E}, \hat{p})$ is a normed linear space over R which contains (E, p) as a linear subspace. In [5] we have shown that the $\aleph_1$-saturation property of ${}^*\mathfrak{M}$ implies that $(\hat{E}, \hat{p})$ is a Banach space. For each internal subspace $F \subset {}^*E$ the space $\hat{F} = \mathrm{st}_p(F \cap p\text{-fin}({}^*E))$ is a closed linear subspace of $\hat{E}$. If F is a *-finite-dimensional subspace of ${}^*E$, then $\hat{F}$ is called a hyper-finite dimensional subspace of $\hat{E}$.

The preceding construction of $\hat{E}$ can be generalized as follows. An element $a \in {}^*E$ is called norm ρ-finite, where ρ is a given positive infinitesimal, whenever ${}^*p(a) \in {}^\rho M_0$. The set of all norm ρ-finite elements of ${}^*E$ will be denoted by p-fin$_\rho({}^*E)$. An element $a \in {}^*E$ is called a ρ-infinitesimal whenever ${}^*p(a) \in {}^\rho M_1$. The set of all ρ-infinitesimals of ${}^*E$ will be denoted by ${}^\rho\mu_p(0)$. Now it is easy to see that the set p-fin$_\rho({}^*E)$ is in a natural way a linear space over ${}^\rho M_0$ and that ${}^\rho\mu_p(0)$ is a linear subspace of p-fin$_\rho({}^*E)$. By ${}^\rho\hat{E}$ we denote the quotient space p-fin$_\rho({}^*E)/{}^\rho\mu_p(0)$ which can be considered as a linear space over ${}^\rho R$. The canonical mapping of p-fin$_\rho({}^*E)$ onto ${}^\rho\hat{E}$ with kernel ${}^\rho\mu_p(0)$ will be denoted by "${}^\rho\mathrm{st}_p$". A norm $\hat{p}_\rho$ can be defined on ${}^\rho\hat{E}$ as follows: If $a \in {}^\rho\hat{E}$ and $x \in p$-fin$_\rho({}^*E)$ such that ${}^\rho\mathrm{st}_p(x) = a$ we set $\hat{p}_\rho(a) = |{}^*p(x)|_\rho$.

We shall now first prove the following theorem.

(3.1) THEOREM. *The space $({}^\rho\hat{E}, \hat{p}_\rho)$ is a nonarchimedean normed linear space over ${}^\rho R$ in the sense of Monna.*

PROOF. It is easy to see that $\hat{p}_\rho(0) = 0$ and $\hat{p}_\rho(a) = 0$ implies ${}^*p(x) = 0$, where ${}^\rho\mathrm{st}_p(x) = a$, and so $x = 0$ implies $a = 0$. Let $t \in {}^\rho R$ and let $a = {}^\rho\mathrm{st}_p(x) \in {}^\rho\hat{E}$ with $x \in p$-fin$_\rho({}^*E)$. Then $\hat{p}_\rho(ta) = |({}^*p(tx))|_\rho = |(|t|{}^*p(x))|_\rho = |t|_\rho \cdot \hat{p}_\rho(a)$. Furthermore, if $a = {}^\rho\mathrm{st}_p(x)$ and $b = {}^\rho\mathrm{st}_p(y)$, where $x, y \in p$-fin$_\rho({}^*E)$, then

$$\hat{p}_\rho(a+b) = |({}^*p(x+y))|_\rho$$

$$\leqq |{}^*\mathrm{p}(\mathrm{x}) + {}^*\mathrm{p}(\mathrm{y})|_\rho \leqq \max(|{}^*p(x)|_\rho, |{}^*p(y)|_\rho) = \max(\hat{p}_\rho(a), \hat{p}_\rho(b)),$$

and the proof is finished.

As in the case of the nonstandard hull of a given normed linear space (E, p), the mapping ${}^\rho\mathrm{st}_p$ imbeds E into a linear subspace of ${}^\rho\hat{E}$. In this case, however, if $a \in E$ and $a \neq 0$, then $\hat{p}_\rho({}^\rho\mathrm{st}_p(a)) = |p(a)|_\rho = 1$. This leads to the following observation that E generates a linear subspace in the quotient space ${}^{\rho}\hat{E}_1/{}^{\rho}\hat{E}_0$ over the residue class field of ${}^{\rho}R$, where ${}^{\rho}\hat{E}_1 = \{a : a \in {}^{\rho}\hat{E} \text{ and } \hat{p}_\rho(a) \leqq 1\}$ and ${}^{\rho}\hat{E}_0 = \{a : a \in {}^{\rho}\hat{E} \text{ and } \hat{p}_\rho(a) < 1\}$.

Let $F \subset {}^*E$ be an internal subspace of ${}^*E$. Then ${}^{\rho}\hat{F} = {}^{\rho}\mathrm{st}_p(F \cap p\text{-fin}_\rho({}^*E))$ with the norm $\hat{p}_\rho$ is a linear subspace of ${}^{\rho}\hat{E}$. If F is a *-finite dimensional subspace of ${}^*E$, then ${}^{\rho}\hat{F}$ is called a hyperfinite dimensional subspace of ${}^{\rho}\hat{E}$.

Given a normed linear space E over a nonarchimedean valuation field K it is natural to ask for whether there exist sufficiently many continuous K-linear functionals on E. It was shown by A. W. Ingleton (see [9], p. 58) that a necessary condition for this to hold is that the norm on E has the ultrametric property. Furthermore, he showed that in order that for a nonarchimedean normed linear space over K the full Hahn–Banach extension theorem holds for K-valued linear functionals it is necessary and sufficient that K is spherically complete. Since the field ${}^{\rho}R$ is spherically complete we may conclude that the Hahn–Banach extension theorem holds for the spaces ${}^{\rho}\hat{E}$.

More generally it was shown (see [9]) that a nonarchimedean normed linear space E has the Hahn–Banach property, that is, for every nonarchimedean normed linear space F over the same field every bounded linear transformation from a linear subspace of F into E can be extended to a linear transformation of F into E with the same norm, if and only if E is spherically complete.

Generalizing Theorem 2.16 we shall now prove the following theorem.

(3.2) THEOREM. *If E is a normed linear space over R and if $\rho > 0$ is an infinitesimal, then the ρ-nonarchimedean normed hull ${}^{\rho}\hat{E}$ of E is spherically complete. Furthermore, if F is an internal subspace of ${}^*E$, then ${}^{\rho}\hat{F}$ is closed and spherically complete. In particular, every hyperfinite dimensional subspace of ${}^{\rho}\hat{E}$ is closed and spherically complete.*

PROOF. In order to prove that ${}^{\rho}\hat{E}$ is spherically complete we have to show that every decreasing sequence of balls has a nondecreasing intersection. To this end, let B_n $(n = 1, 2, \cdots)$ be a decreasing sequence of balls of radius r_n $(n = 1, 2, \cdots)$, and let $r = \inf r_n$. If $r = r_n$ for some n, then there is nothing to prove. If this is not the case, then there is a sequence $\{a_n\}$ $(n = 1, 2, \cdots)$ such that $a_n \in B_n$ and $a_n \notin B_{n+1}$. If $k < l < m$, then $a_m \in B_m \subset B_l$ and $a_l \in B_l$, and so $\hat{p}(a_m - a_l) < r_l$. Since $a_k \notin B_l$ we have also $\hat{p}_\rho(a_k - a_l) > r_l$, which implies that $\hat{p}_\rho(a_m - a_l) < \hat{p}_\rho(a_k - a_l)$. Hence, $\hat{p}_\rho(a_k - a_l) = \hat{p}_\rho(a_{l+1} - a_l)$ for all l and $k \geqq l$. Let $x_k \in {}^*E$ be such that ${}^{\rho}\mathrm{st}_p(x_k) = a_k$ $(k = 1, 2, \cdots)$, then

$$\exp(-\log_\rho {}^*p(x_k - x_l)) - \exp(-\log_\rho {}^*p(x_{l+1} - x_l)) =_1 0$$

for all $k \geqq l$. Hence, for each $l = 1, 2, \cdots$, there exists an infinitesimal $0 < h_l \in M_1$ such that $|\log_\rho(^*p(x_k - x_l)) - \log_\rho(^*p(x_{l+1} - x_l))| \leq h_l$ for all $k \geqq l$. Let us now assume that the sequence $\{x_k\}$ is extended into $^*E$ over $^*N$ into an internal sequence. If, incidentally, $x_k \in F$ $(k = 1, 2, \cdots)$ where F is internal, then the extended sequence may be assumed without loss of generality to be extended with values in F. Then as in the proof of Theorem 2.16 we may conclude that there is an infinitely large natural number ω_0 and a positive infinitesimal $h_0 > 0$ such that for all $n = 1, 2, \cdots$ $|\log_\rho(^*p(x_{\omega_0} - x_m)) - \log_\rho(^*p(x_{n+1} - x_n))| \leqq h_0$. Then $^*p(x_{\omega_0}) \leqq {}^*p(x_{\omega_0} - x_n) + {}^*p(x_n)$ shows that $x_{\omega_0} \in p\text{-fin}_\rho(^*E)$ and hence, $a_0 = {}^\rho\mathrm{st}_\rho(x_{\omega_0}) \in {}^\rho\hat{E}$, and $\hat{p}_\rho(a_0 - a_n) = \hat{p}_\rho(a_{n+1} - a_n)$ for all $n = 1, 2, \cdots$. Hence, $\hat{p}_\rho(a_0 - a_n) \leqq r_n$ for all $n = 1, 2, \cdots$; and the proof is finished.

Remark. It is easy to see that every bounded linear functional $x' \in E'$ on E has a natural extension to the space ${}^\rho\hat{E}$. To this end we only have to observe that x' being bounded it satisfies a Lipschitz condition of order one with respect to the norm on E. Hence, its extension to $^*E$, which we shall again denote by x', satisfies the property that if $x \in p\text{-fin}_\rho(^*E)$, then $|\langle x, x'\rangle| \in {}^\rho M_0$, and if $x, y \in p\text{-fin}_\rho(^*E)$ satisfy $\|x - y\| =_\rho 0$, then $\langle x, x'\rangle =_\rho \langle y, x'\rangle$. From this we conclude that on the quotient space ${}^\rho\hat{E}$ the functional x' defines a bounded linear functional of ${}^\rho\hat{E}$ into ${}^\rho R$ and defines an imbedding of the Banach dual E' of E into the dual space of ${}^\rho\hat{E}$. The properties of this imbedding and related questions will be discussed in another paper.

References

1. I. Kaplansky, *Maximal fields with valuations*, Duke Math. J. **9** (1942), 303–321.
2. W. Krull, *Allgemeine Bewertungstheorie*, Crelle Journ. für die Mathematik **167** (1932), 160–196.
3. D. Laugwitz, *Tullio Levi-Cività's work on nonarchimedean structures*, Accademia Nazionali dei Lincei, Roma, 1975.
4. T. Levi-Cività, *Sugli infiniti ed infinitesimi attuali quali elementi analitia*, Atti Inst. Veneto Sci., Lett. Arti, ser. 7a, **4** (1892–1893), 1765–1815.
5. W. A. J. Luxemburg, *A new approach to the theory of monads*, in *Applications of Model Theory to Algebra, Analysis and Probability*, Proc. of the 1967 Pasadena Symposium on Nonstandard Analysis, Holt-Rinehart and Winston, New York, 1969.
6. W. A. J. Luxemburg, *What is nonstandard analysis?* Amer. Math. Monthly **80** (1973), 36–67.
7. W. A. J. Luxemburg, *Nichtstandard-Zahlsysteme und die Begründung des Leibnizschen Infinitesimalkalküls*, in Jahrbuch Ueberblicke Mathematik (1975) 31–44, Bibliographisches Institut AG, Manheim, Wien, Zürich, 1975.
8. W. A. J. Luxemburg and K. Stroyan, *Introduction to Nonstandard Analysis*, Academic Press, New York, 1976.
9. A. F. Monna, *Analyse Non-Archimédienne*, in *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 56, Springer-Verlag, Berlin–Heidelberg–New York, 1970.

10. A. Robinson, *Non-standard Analysis, Studies in Logic*, North-Holland Publ. Co., Amsterdam, 1966 (second printing 1974).

11. A. Robinson, *Function theory on some nonarchimedean fields*, Amer. Math. Monthly **80** (1973), 87–109.

12. A. Robinson and A. H. Lightstone, *Non-Archimedean Fields and Asymptotic Expansions*, North-Holland Mathematical Library, Vol. 13 (1975), North Holland Publ. Co., Amsterdam–Oxford/American Elsevier Publ. Co., Inc., New York, 1975.

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